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Long strings in two dimensional string theory and non-singlets in the matrix model

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Abstract: We consider two dimensional string backgrounds. We discuss the physics of long strings that come from infinity. These are related to non-singlets in the dual matrix model description.

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Contents

1. Introduction

The main motivation for this paper is to get some insights about the lorentzian physics of the two dimensional black hole [1]. A proposal for the matrix model dual of the euclidean black hole was made by Kazakov, Kostov and Kutasov in [2]. These authors proposed a matrix model description which involves adding Wilson lines for the ordinary gauged matrix model that describes the two dimensional string background [3]. This implies that we need to understand the matrix model in its non-singlet sector. In this paper we study aspects of the physics of the matrix model in its non-singlet sector. While we will not give a picture for the two dimensional black hole, the remarks in this paper might be helpful in this quest. Further work on this correspondence includes [4 − 17].

We will show that the double scaling limit of the matrix model in its simplest non-trivial representation, the adjoint representation, is related to a two dimensional string background with a folded string that comes in from the weak coupling region and goes back to the weak coupling region. Such a string is not static. Indeed, we get a time dependent configuration. In the double scaling limit, the system does not have a well defined ground state but it has interesting time-dependent solutions. An interesting observable is the scattering amplitude for a folded string that comes from the weak coupling region and goes back to the weak coupling region.[1] This phase can be computed exactly at tree level in string theory using formulas in [20]. We compare this to the matrix model computation. We reduce the matrix model computation to a rather simple looking eigenvalue problem. Though we were not able to solve completely this eigenvalue problem we could show that in the two asymptotic limits of high and low energies, the answer matches with the corresponding exact expressions in string theory. Previous work on the non-singlet sector includes [21, 22]. This picture of the non-singlet sector is important for understanding the physical meaning of the results of computations done via T-duality or the euclidean worldsheet theory.

We consider the same problem also in the type 0A/0B superstring cases, where a new feature arises. There are two answers that depend on the treatment of a particular fermion zero mode.

This paper is organized as follows. In section 2 we discuss the physics of long strings in two dimensional string theory. In section 3 we consider non-singlets in the hermitian matrix model. We first concentrate on the adjoint representation and then we discuss some features for general representations. In section 4 we consider non-singlets in the complex matrix model that is dual to 0A superstring theory. We end with a discussion.

2. Long strings in two dimensional string theory

2.1 Classical strings in two dimensions

Let us first consider a simple problem [18]. Suppose we have a classical string on a two dimensional spacetime parametrized by X^0, ϕ. We choose a gauge with $\partial_+ X^0 = \partial_- X^0 = 1$ with $\partial_\pm = \partial_\tau \pm \partial_\sigma$, where τ, σ are the worldsheet coordinates. The classical equations then

[1]Folded strings have also been previously studied in [18, 19].

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reduce to $\partial_+\partial_-\phi = 0$ and the Virasoro constraints

$$-1 + (\partial_+\phi)^2 = 0\,, \qquad -1 + (\partial_-\phi)^2 = 0\,. \tag{2.1}$$

Then we conclude that $\partial_+\phi = \pm 1$. The first derivative does not have to be continuous. So the most general solution has the form $\phi = \phi_+(\sigma^+) + \phi_-(\sigma^-)$ where ϕ_+ and ϕ_- are periodic functions with the same period and with derivative piece-wise equal to ± 1. A simple example would be

$$\phi_+(s) \equiv \phi_-(-s) \equiv |s - L/2| \qquad \text{for } 0 < s < L \tag{2.2}$$

and defined in a periodic fashion outside this interval. This describes a pulsating string. At $X^0 = 0$ the string is folded and stretched to its maximum length L from $\phi = 0$ to $\phi = L$. At later times the two folds start moving towards each other at the speed of light, they cross at $X^0 = L/2$ and they end end up with a configuration similar to the original one at $X^0 = L$. See figure 1. We can think of the whole configuration as two massless particles joined by a string. In fact, the energy levels that we get are similar to those we get for mesons in 2d QCD with massless quarks at high excitations levels [23].[2] Here we looked at a simple solution, but the string can have many folds, see [18, 19] for further discussion.

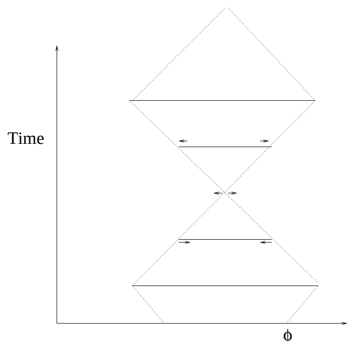

Figure 1: Two dimensional pulsating string. The ends of the string move at the speed of light.

2.2 Long strings with a linear dilaton

Now let us consider the situation with linear dilaton, $\Phi = Q\phi$. Then the Virasoro constrains in (2.1) are replaced by[3]

$$-1 + (\partial_+\phi)^2 - Q\partial_+^2\phi = 0\,, \qquad -1 + (\partial_-\phi)^2 - Q\partial_-^2\phi = 0\,. \tag{2.3}$$

Solving these equations we find

$$\phi = \phi_0 - Q\log\cosh\frac{\sigma^+}{Q} - Q\log\cosh\frac{\sigma^-}{Q} = \phi_0' - Q\log\left(\cosh\frac{\tau}{Q} + \cosh\frac{\sigma}{Q}\right)\,. \tag{2.4}$$

This is the most general solution after we allow shifts of τ and σ. The solution corresponding to the usual pointlike string, where $\phi = \pm\tau$, can be viewed as a limit of this solution. Except for this special limit, the solution (2.4) is not periodic. It represents a folded string stretched to minus infinity in the ϕ direction. The tip of the string moves from the weakly coupled region into strong coupling and back into the weakly coupled region. One minor difference with the strings discussed in the previous section is that in this case the tip of the string looks a bit more like a massive particle, with a mass of order QM_s. After a few string times the tip of the string is accelerated to nearly the speed of light so that its mass

[2] In 2d QCD we have an open string rather than a folded string but the form of the solutions is essentially identical.

[3] Throughout this paper we set $\alpha' = 1$ for the bosonic string and $\alpha' = \frac{1}{2}$ for the superstring.

becomes unimportant. Note that we do not have oscillating strings. This agrees with the fact that, for $Q = 2$, there are no closed string states in the linear dilaton region other than the massless "tachyon".

We can think of this string as a particle (the tip of the string) with a linear potential (provided by the string tension). This enables us to use a WKB approximation for the wavefunction. The particle momentum and wavefunction are given by

$$p = \pm 2 \frac{1}{2\pi}(\phi_m - \phi)\,, \qquad \psi \sim e^{i\int d\phi p} \sim e^{\pm i \frac{1}{2\pi}(\phi - \phi_m)^2} \tag{2.5}$$

where we used the relativistic dispersion relation that is valid for $p \gg 1$. In (2.5) we also used that the force is given by twice the tension since the string is folded. Here ϕ_m is just an integration constant which is the value of ϕ at which the tip of the string bounces back. This is then related to the energy of the string through

$$E = 2\frac{1}{2\pi}(\phi_m - \phi_c)\,, \qquad \epsilon \equiv E - E_{div} = \frac{1}{\pi}\phi_m\,, \qquad E_{div} = -\frac{1}{2\pi}\phi_c \tag{2.6}$$

where ϕ_c is some cutoff at large negative values. We have defined the finite energy ϵ by subtracting a divergent piece. This divergence arises because the string is stretching all the way to $\phi \to -\infty$.

In the $\phi \to -\infty$ region, (2.5) will be a correct approximation for the full wavefunction. We will check this statement later both on the matrix model side and in the exact worldsheet analysis. So we can define the scattering amplitude through[4]

$$\psi = e^{-i\frac{1}{2\pi}(\phi-\phi_m)^2} - e^{i\delta} e^{i\frac{1}{2\pi}(\phi-\phi_m)^2}\,. \tag{2.7}$$

In this definition, ϕ_m is just a constant related to the energy ϵ of the string through (2.6). In the linear dilaton region we see that the phase (2.7) is independent of the energy since it depends only on the behavior of the wavefunction for $\phi \sim \phi_m$ and ϕ_m drops out of this problem. In other words, the essential physics of this problem is translation invariant and it does not matter precisely where the string bounces back. We will define the constant relative phase between the two terms in (2.7) in such a way that this constant is zero

$$e^{i\delta} = 1\,. \tag{2.8}$$

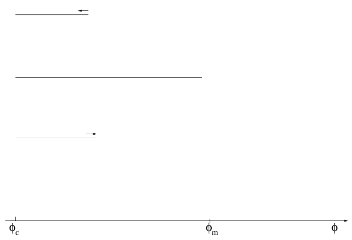

Figure 2: Long string coming in from $\phi = -\infty$ and bouncing back. The vertical direction is time and we show snapshots at different times. The strings stretch from $\phi = -\infty$, or a large negative cutoff value ϕ_c. The tip comes from the left and, after it reaches the largest value of $\phi = \phi_m$, it starts moving back to the left.

2.3 Strings in (Liouville)×(Time)

Let us now consider the Liouville case. The general solution of the classical Liouville equation can be written as

$$\mu e^{2b\phi} = \frac{1}{[\sum_i f_i^+(b\sigma^+) f_i^-(b\sigma^-)]^2} \tag{2.9}$$

[4] This is a somewhat non-standard definition of the phase. The relation to the more standard definition can be found in appendix A. This is a detail most readers will want to ignore in a first reading.

where f_i obey the conditions

$$\partial^2 f_i - f_i = 0\,, \qquad \epsilon_{ij} f_i \partial f_j = 1\,. \tag{2.10}$$

The first comes from (2.3) , with $Q = 1/b + b \sim 1/b$, where we assume that b is small which is the classical limit. The second equation in (2.10) comes from the Liouville equation. We then conclude that the most general solution is

$$\mu e^{2b\phi} = \frac{4}{(Ae^{b\sigma^+ + b\sigma^-} + Be^{b\sigma^+ - b\sigma^-} + Ce^{-b\sigma^+ + b\sigma^-} + De^{-b\sigma^+ - b\sigma^-})^2} \tag{2.11}$$

where $AD - BC = 1$. Some particular interesting cases are $A = D = 1$ $B = C = 0$. In this case we get

$$\mu e^{2b\phi} = \frac{1}{\cosh^2 b\tau} \tag{2.12}$$

where $\tau = \sigma^+ + \sigma^-$. In this case we can compactify the σ direction. This is the standard solution representing a small string coming in from infinity. The spacetime energy of the string is related to the size of the σ circle. If we rescale this circle so that it has standard length 2π, then we also have to shift ϕ, since scale transformations also act on the Liouville field. Then we see that the maximum value of ϕ is related to the energy of the incoming particle [3]. Another interesting solution is $a = d = \cosh\gamma$ and $b = c = \sinh\gamma$ then the solution is

$$\mu e^{2b\phi} = \frac{1}{(\cosh\gamma\cosh b\tau + \sinh\gamma\cosh b\sigma)^2}\,. \tag{2.13}$$

This represents the string coming from infinity and going back to infinity. For $\alpha = 0$ we get (2.12) (with $Q \sim 1/b$). In the limit $\alpha \to \infty$ we get back (2.4). In this case the string "bounces back" before it gets to the region where the Liouville potential is important.

2.4 Exact worldsheet results for the scattering of long strings

An interesting observable for these strings is the scattering amplitude for the strings to go in and come back to infinity. At the classical level (in the g_s expansion) this is just a phase. We can compute this phase exactly in α' by using the results for the disk two point functions in [20]. So we introduce an FZZT brane [20, 24] that is extended in ϕ but with $\mu_B \gg \sqrt{\mu}$ so that it dissolves at a very large negative value of $\phi \sim -\log\mu_B$. Then we consider an open string living on this D-brane. We send in the open string towards the strong coupling region with high energy E_0. When the open string reaches the region $\phi \sim -\log\mu_B$, its ends get stuck in this region while the bulk of the open string stretches as in the solutions we considered above. Eventually the energy of the stretched string becomes equal to the initial energy. At this point the string will stretch between $\phi_c \equiv -\log\mu_B$ to $\phi_m \sim E_0\pi - \log\mu_B$ (see the related formula (2.6)). If we take a limit with

$$E_0 \to \infty\,, \qquad \mu_B \to \infty\,, \qquad \epsilon = E_0 - \frac{1}{\pi}\log\mu_B = \text{finite} \tag{2.14}$$

then we get the problem that we are interested in. Namely, we have an infinitely stretched string that comes in from infinity and goes back to infinity. We are introducing the FZZT branes and the open strings on them in order to be able to use the formulas in [20] to compute the amplitude we are interested in. Starting with the open string two point function in [20] and performing the scaling limit in (2.14) , we obtain the scattering phase (see appendix A for details)

$$\delta(\epsilon) = -\int_{-\infty}^{\hat{\epsilon}} d\epsilon' \left(\frac{\pi\epsilon'}{\tanh \pi\epsilon'} + \pi\epsilon' \right) \hat{\epsilon} \equiv \epsilon + \frac{1}{2\pi} \log \mu \,. \tag{2.15}$$

The answer is a function of $\hat{\epsilon}$. This can be understood as follows. The Liouville potential becomes important at $\phi_L \sim -\frac{1}{2}\log\mu$. The energy ϵ is related to the point were the string would stretch to a maximum in the absence of Liouville potential by (2.6). So the amplitude is just a function of $\phi_m - \phi_L$, which is proportional to $\hat{\epsilon}$. We see that the phase $\delta(\epsilon) \to 0$ for $\epsilon \to -\infty$, so that we recover the linear dilaton result (2.8). In fact, the asymptotic behavior of this phase is

$$\delta(\epsilon) \sim \left(\hat{\epsilon} - \frac{1}{2\pi} \right) e^{2\pi\hat{\epsilon}} = \left(\hat{\epsilon} - \frac{1}{2\pi} \right) \mu e^{2\pi\epsilon} \,, \qquad \text{for } \epsilon \ll \log \mu \,. \tag{2.16}$$

We see that the leading corrections to the $\epsilon \to -\infty$ answer has the form we expect. To leading order we get that the phase is zero, in agreement with (2.8) , and the first correction can be interpreted as resulting from the insertion of a cosmological constant operator. Indeed, in this case the string bounces back before it gets to the region where the Liouville potential is important. On the other hand, the high energy behavior is

$$\delta(\epsilon) \sim -\pi\hat{\epsilon}^2 + \left(\hat{\epsilon} + \frac{1}{2\pi} \right) e^{-2\pi\hat{\epsilon}} \,, \qquad \text{for } \epsilon \gg \log \mu \,. \tag{2.17}$$

The leading result in (2.17) can also be easily understood. We can think of the Liouville potential as a sharp wall at $\phi_L = -\frac{1}{2}\log\mu$. This imposes a boundary condition on the wavefunction at this point, say $\psi(\phi_L) = 0$. From (2.7) we see that this implies that

$$\delta \sim -\frac{1}{\pi}(\phi_L - \phi_m)^2 = -\pi\hat{\epsilon}^2 \tag{2.18}$$

which is just the leading behaviour of (2.17). Note that if the energy gets extremely large, $\epsilon \sim \mu$, then we expect that this approximation would fail, since that is enough energy to create a ZZ brane. But for large μ there is a large range of energies, $\mu \gg \epsilon \gg \log\mu$, where we can trust (2.17).

2.5 Worldsheet formulas for the two dimensional type 0 superstring

For the type 0 superstring formulas we set $\alpha' = \frac{1}{2}$. By starting with the exact Liouville formulas in [25] and taking a limit where $\mu_B \to \infty$ we find two possible results (see appendix A)

$$\delta_+(\epsilon) = -2\int_{-\infty}^{\frac{\hat{\epsilon}}{2}} d\epsilon' \pi\epsilon' (\tanh \epsilon'\pi + 1) \,, \qquad \delta_-(\epsilon) = -2\int_{-\infty}^{\frac{\hat{\epsilon}}{2}} d\epsilon' \pi\epsilon' \left(\frac{1}{\tanh \epsilon'\pi} + 1 \right) \tag{2.19}$$

with

$$\hat{\epsilon} \equiv \epsilon + \frac{1}{\pi} \log \mu \,. \tag{2.20}$$

The first thing we need to understand is why we get two different results. Let us first look at the behavior of (2.19) for $\hat{\epsilon} \ll 0$. To leading order both results give us that $\delta = 0$, which agrees with the result one expects in the linear dilaton region. The first correction is

$$\delta_{\pm} \sim \mp \left(\hat{\epsilon} - \frac{1}{\pi} \right) e^{\pi \hat{\epsilon}} = \mp (\hat{\epsilon} - \frac{1}{\pi}) \mu e^{\pi \epsilon} \,, \qquad \hat{\epsilon} \ll 0 \,. \tag{2.21}$$

So the only difference in the two results is the overall sign. In the linear dilaton region the field ϕ is given by (2.4). In a gauge where the worldsheet fermion $\psi^0 = 0$ we also need to set to zero the supercharge in the ϕ direction. This will lead to the equation

$$\psi_+^\phi \partial_+ \phi - Q \partial_+ \psi_+^\phi = 0 \tag{2.22}$$

and a similar equation for left movers. The only solution to (2.22) is

$$\psi_+^\phi = \frac{1}{\cosh \frac{\sigma^+}{Q}} \epsilon_0 \tag{2.23}$$

where ϵ_0 is a constant. It can be checked that this mode is in the cohomology, namely it is not given by acting with the supercharge on the original configuration (2.4). Upon quantization, the left and right moving modes $\epsilon_0^{l,r}$ together lead to a two dimensional Hilbert space ($\epsilon_0^{l,r} \sim \sigma^{1,2}$, where σ^i are Pauli matrices). The cosmological constant term $\mu \psi_+^\phi \psi_-^\phi e^{2\phi}$ reduces to $\sigma^3 \mu e^{2\phi}$ when it is projected onto the two dimensional space Hilbert space. So we see that we will get two different answers. When the scattering occurs in the large negative ϕ region the leading correction comes from inserting just one cosmological constant operator and this leads to the two possible signs in (2.21).

The results in (2.19) were actually obtained by scattering NS open strings on an FZZT brane with $\eta = \pm 1$. Recall that η is a sign which relates the left moving to the right moving supercharge at the boundary of the worldsheet. We are using the conventions of [25] where the ZZ brane has $\eta = -1$. The sign of η is telling us the relative sign between the left and right moving fermion. Since the cosmological constant term goes as $\mu \psi_L \psi_R e^{2\phi}$, the relative sign between the left and right moving fermion determines the sign of the first correction. The precise boundary conditions at the end of the string should impose that only one of the two states associated to the fermion zero modes we discussed above are kept. We have not worked out the details of how this happens, but it follows from the results in [25].

The results are basically the same in type 0A and 0B theories, though some of the detailed properties of the FZZT branes are different.

We can also consider the scattering of Ramond open strings ending on FZZT branes with different η. In the $\mu_B \to \infty$ limit that we are considering we find results as in (2.19).

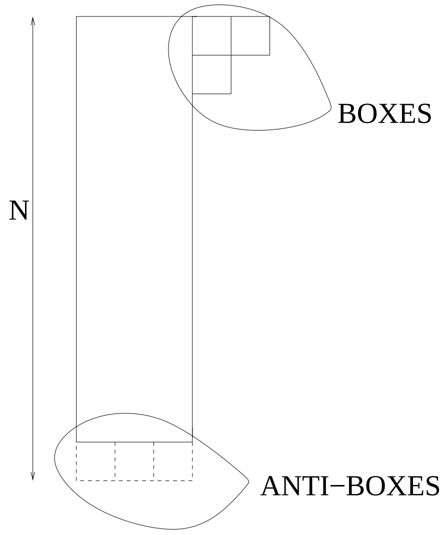

Figure 3: Young diagram of a possible representation. In this case we have three boxes and three anti-boxes.

3. Non-singlets in the matrix model

In this section we consider the SU(N) gauged quantum mechanics of a single hermitian matrix Φ with the insertion of a Wilson line in the representation $\mathcal{R}$

$$Z = \int \mathcal{D}\Phi\mathcal{D}A e^{\int dt \, \mathrm{Tr}[(D_0\Phi)^2 - V(\Phi)]} \, \mathrm{Tr}_{\mathcal{R}} \, e^{i\int A} \, . \tag{3.1}$$

When we analyze this model we can equivalently set $A_0 = 0$ and consider only states in the Hilbert space of the matrix Φ that transform in the representation $\mathcal{R}^\dagger$. From now on we will replace $\mathcal{R} \to \mathcal{R}^\dagger$ for notational convenience. This problem was considered in [26, 21, 22, 27–29].[5] Of course, the representation $\mathcal{R}$ should be such that it can be constructed by taking multiple products of the adjoint representation. So the representation should be invariant under the center of SU(N). We are going to be interested in taking the large-N limit and we want to keep the representation $\mathcal{R}$ "fixed". In this limit it is useful to think of the Young diagram for representation as made up with some boxes and "anti-boxes", see figure 3. We have the constraint that the number of boxes and anti-boxes should be equal. But the Young diagram for the boxes need not be equal to the Young diagram for anti-boxes. In other words the representation need not be self conjugate [22].[6]

It is convenient to diagonalize the matrix $\Phi = \mathrm{diag}(\lambda_1, \cdots, \lambda_M)$. This gauge choice breaks SU(N) to U(1)$^{N-1}$. Then we can write the wavefunction as $\psi_\alpha(\lambda_i)$ where α is an index in the representation $\mathcal{R}$, which labels the states with zero weight in this representation, namely states with zero U(1)$^{N-1}$ charges. The restriction to zero weight states comes

[5] One minor difference with some of these papers is that here we do not include a factor proportional to the dimension of the representation in the partition function.

[6] Only self conjugate representations were considered in [21].

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from integrating out the diagonal part of the gauge field A_0. The hamiltonian is

$$H = \left[\sum_{i=1}^{n} -\frac{1}{2}\frac{\partial^2}{\partial\lambda_i^2} - \frac{1}{2}\lambda_i^2 + \frac{1}{2}\sum_{i\neq j}\frac{T_{ij}^{\mathcal{R}}T_{ji}^{\mathcal{R}}}{(\lambda_i-\lambda_j)^2}\right] P_0 \tag{3.2}$$

where P_0 is the projector on to states in the representation $\mathcal{R}$ with zero $\mathrm{U}(1)^{N-1}$ charges. From now on we will drop it. $T_{ij}^{\mathcal{R}}$ are the matrices corresponding to the ij generator of $\mathrm{SU}(N)$ in the representation $\mathcal{R}$.

3.1 The adjoint representation

This case was considered for a general potential in [26]. Here we review their procedure and we will further explore it in the case of the inverted harmonic oscillator, where some of its features were discussed in [21]. The adjoint representation is just given by a matrix W_i^j. The states with zero weight are diagonal matrices $W_i^j = w_i\delta_i^j$, with $\sum_i w_i = 0$. We see that the projection to zero weight states reduced the number of states from N^2-1 to $N-1$.

We will now treat the interaction in (3.2) perturbatively. To zeroth order we neglect it and the hamiltonian is just that of free fermions, with the usual ground state. This ground state is degenerate since we can choose any state W in the representation. To first order the interaction term in (3.2) removes this degeneracy. We can think of the representation part of the wavefunction as given by a state vector $|W\rangle = \sum_i w_i|i\rangle \times |i\rangle$ Then the interaction part of the hamiltonian acts as

$$H_{int}|W\rangle = \sum_{i=1}^{N}\left(\sum_{j=1,\ j\neq i}^{N}\frac{(w_i-w_j)}{(\lambda_i-\lambda_j)^2}\right)\Big|i\Big\rangle \times \Big|i\Big\rangle\,. \tag{3.3}$$

The interaction hamiltonian is positive definite. This is most easily seen by computing $\langle W|H_{int}|W\rangle$. After introducing the eigenvalue density in the standard way and writing $w_i \to w(\lambda)$ we end up with the following eigenvalue problem

$$Ew(\lambda) = \int d\lambda' \rho(\lambda')\frac{w(\lambda)-w(\lambda')}{(\lambda-\lambda')^2}\,. \tag{3.4}$$

It looks like the lowest energy state is $w(\lambda)$ =constant. However, this state is not in the adjoint representation which is constrained to have $\int d\lambda\rho(\lambda)w(\lambda) = 0$. Some approximate solutions to this problem can be found in [26].

Here we are interested in understanding the inverted Harmonic oscillator potential. In this case it is more convenient to introduce a slightly different variable

$$h(\lambda) = w(\lambda)\rho(\lambda)\,. \tag{3.5}$$

In terms of this variable the eigenvalue problem we need to solve becomes

$$\begin{aligned} Eh(\lambda) &= -\rho(\lambda)\int d\lambda'\frac{1}{(\lambda-\lambda')^2}h(\lambda') + v(\lambda)h(\lambda) \\ v(\lambda) &\equiv \int d\lambda'\frac{\rho(\lambda')}{(\lambda-\lambda')^2}1 = \int\frac{d\lambda}{\rho(\lambda)}h^2(\lambda) \end{aligned} \tag{3.6}$$

where we have also included the form of the normalization condition.[7] In all integrals we should take the principal value, which means that we subtract all divergent terms when $\lambda' \to \lambda$. The first term (3.6) can be viewed as a kinetic term and the second as a potential. This potential contains a constant divergent term plus a finite term. Let us first understand the problem in the asymptotic region where $\lambda \gg \sqrt{\mu}$, we will later discuss the more general case. In the asymptotic region we can approximate $\rho(\lambda) = \frac{1}{\pi}\lambda$ and define $\tau \equiv \log \lambda$. We obtain

$$v(\lambda) = \frac{1}{\pi} \int_0^{\lambda_c} d\lambda' \frac{\lambda'}{(\lambda' - \lambda)^2} \sim \frac{1}{\pi}(\tau_c - \tau - 1) \tag{3.7}$$

where τ_c is some large cutoff value of τ. The divergence comes from the eigenvalues that are far away, at large λ'. The region $\lambda' \sim \lambda$ does not lead to a divergence because we are using a principal value prescription. So, up to the divergent term we find a linear potential that is pushing the particle to the large τ region. Similarly we can study the kinetic term in the asymptotic region and we find that the normalization condition and the kinetic term, K, become

$$\int d\tau h^2(\tau) = 1\,, \qquad (Kh)(\tau) = -\frac{1}{\pi} \int d\tau' \frac{h(\tau')}{(2\sinh(\frac{\tau-\tau'}{2}))^2} \tag{3.8}$$

Since the kinetic term became translational invariant it is reasonable to solve the problem using plane waves. Computing the kinetic term in Fourier space we find

$$K(k) = -\frac{1}{\pi} \int_{-\infty}^{\infty} d\tau \frac{\cos k\tau}{4\sinh^2 \frac{\tau}{2}} = \frac{k}{\tanh(\pi k)} \tag{3.9}$$

where we take the principal part of the integral (i.e. we disregard terms that diverge as $\tau \to 0$). At large k we have a relativistic relation

$$E \sim |k| \qquad \text{for } |k| \gg 1\,. \tag{3.10}$$

So we see that the Hilbert space spanned by the vector w_i can be thought of as the Hilbert space of the particle that lives at the tip of the string. The string provides the linear potential (3.7). It is interesting to see how the dynamics of this extra particle is generated out of the interaction with the background fermions. The coefficients in (3.7) and (3.9) match precisely with our expectations based on the string theory discussion after we identify $\tau \sim -\phi$, where ϕ is the Liouville direction. This is a standard identification valid for large values of τ [3]. In fact, the divergence in (3.7) is related to the fact that the string is stretched all the way to $\phi = -\infty$. We can solve the complete quantum scattering problem in this asymptotic region and one finds a result that agrees with (2.8), see appendix B.

A few remarks are in order. First, note that, after subtracting the infinite additive constant (3.7) , the spectrum is *unbounded below.*[8] This has some important consequences

[7] Note that $v(\lambda)$ in (3.6) is not the same as $V(\lambda)$ in the original matrix model lagrangian.

[8] The authors of [21] stated that the spectrum has a gap of order $\log \Lambda/\mu$ (which is the infinite subtraction we are performing) and then a continuum spectrum similar to that of the free fermions. What is important is that this continuous spectrum is unbounded below.

when we examine the physics of this model. Second, the infinite constant that we are subtracting behaves as

$$E_{div} \sim \frac{1}{\pi}\tau_c = -\frac{1}{\pi}\phi_c\,. \tag{3.11}$$

There is precise agreement in the coefficient of this divergent term between the matrix model and string theory.

Now let us come back to the general problem in (3.6) and include the μ dependence. It is convenient to define the τ variable through $\lambda = \sqrt{2\mu}\cosh\tau$. Then $\rho(\lambda) = \sqrt{2\mu}\sinh\tau$. We define $h(-\tau) \equiv -h(\tau)$ for $\tau > 0$. The sign in this definition is motivated by the fact that we expect a boundary condition $h(\tau = 0) = 0$, since at $\tau = 0$ we run out of fermions which could carry the adjoint indices. With this definition of h for negative values of τ we can extend the range of the integral over the whole τ axis and obtain a simple expression for the kinetic term. Namely, the same as the one we had in (3.8). We also subtract the divergent piece of the energy. The eigenvalue problem then becomes

$$\hat\epsilon h(\tau) = -\frac{1}{\pi}\int_{-\infty}^{\infty} d\tau' \frac{h(\tau')}{4\sinh^2\frac{\tau-\tau'}{2}} + \hat v(\tau)h(\tau)\hat v(\tau) = -\frac{1}{\pi}\frac{\tau}{\tanh\tau} \tag{3.12}$$

with

$$\hat\epsilon = E - \frac{1}{\pi}\tau_c + \frac{1}{\pi} = E + \frac{1}{\pi}\log\lambda_c + \frac{1}{2\pi}\log\mu + \text{const} = \epsilon + \frac{1}{2\pi}\log\mu \tag{3.13}$$

where ϵ should be thought of as the renormalized energy since it is the one defined with a μ independent subtraction procedure. We have absorbed some constants in this subtraction in order to make formulas look simpler. Note that μ has completely disappeared from (3.12) , which agrees with the μ dependence of the string theory result (2.15). The energy ϵ defined in (3.13) agrees with the renormalized energy defined in (2.6) or (2.14) , up to an additive numerical constant which we have fixed by comparing the matrix model and the string theory answers. In summary, the energy ϵ which appears in (3.13) should be directly identified with ϵ in (2.15).

We have been unable to solve the problem of computing the phase directly from (3.12). Nevertheless we conjecture that the scattering phase that comes out of it is given by

$$\delta(\hat\epsilon) = -\int_{-\infty}^{\hat\epsilon}\left(\frac{\pi\epsilon'}{\tanh\pi\epsilon'} + \pi\epsilon'\right) \tag{3.14}$$

with a definition of the scattering phase analogous to (2.7). We have computed explicitly the two asymptotic limits, $\epsilon \to \pm\infty$, as well as the first exponential subleading correction in each of these asymptotic regions and checked that they match (2.16) (2.17). See appendix B for details. It seems very likely that there is a simple way to solve this problem exactly to derive (3.14).

3.2 The vortex anti-vortex correlator

So far we have been discussing the lorentzian theory. Let us now consider the euclidean theory with euclidean time compactified on a circle of radius R. Thus we compute the thermal partition function for the matrix model in the adjoint representation. On the

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worldsheet it is easy to compute it. It is given by the two point function for two winding strings. Let us first perform an approximate qualitative analysis. The Liouville correlation function involves an integral of the from

$$\int d\phi e^{-\mu e^{2\phi}} e^{2(2-R)\phi} e^{-4\phi} . \tag{3.15}$$

The last term comes from the dilaton. We see that this diverges at weak coupling as $\int d\phi e^{-2R\phi}$. This divergence, which has a positive coefficient, is related to the energy of the stretched string. In other words, expect a divergence of the form $\int d\epsilon e^{-(2\pi R)\epsilon}$ for $\epsilon \to -\infty$. This divergence is related to the fact that, after the subtraction, the spectrum of the matrix model hamiltonian is unbounded below. This divergence is independent of μ. If we concentrate only on μ dependent terms, then we can get a finite answer. We can compute this by taking a derivative with respect to μ of (3.15). This then gives a convergent integral for $R < 1$

$$\partial_\mu \langle 2pt\rangle \sim -\int d\phi e^{-\mu e^{2\phi}} e^{2\phi(1-R)} \sim -\mu^{R-1}\Gamma(1-R) . \tag{3.16}$$

This has just been an approximation to the true answer. The correct answer is computed using the exact expression for the Liouville three point function in [30]. After taking the limit $b \to 1$ in the formulas in [30] we obtain

$$\partial_\mu \langle 2pt\rangle_{Liouville} = -\langle 3pt\rangle_{Liouville} = -\frac{\mu^{1-R}}{\pi} R^2 \frac{\Gamma(-R)^2}{\Gamma(R)^2} . \tag{3.17}$$

This implies that the two point function in the full string theory is then given by

$$\langle 2pt\rangle = -2R^2\mu^R \frac{\Gamma(-R)^2}{\Gamma(R)^2} \sim \frac{Z_{\rm adj}}{Z_{\rm singlet}} \tag{3.18}$$

up to a (positive) numerical constant. Going from (3.17) to (3.18) we loose one factor of R from integrating (3.17) and we gain one factor of $(2\pi R)$ from the X path integral. Notice that this formula is consistent with the statement that the free energy in the adjoint differs from the singlet free energy by a logarithmic term

$$F_{adj} \sim F_{sing} - \frac{1}{2\pi} \log \mu . \tag{3.19}$$

Note that the coefficient of the logarithmic term agrees precisely with the matrix model,[9] due to the agreement in the divergent parts of the energy that we checked above, see (3.11). The sign in (3.18) seems puzzling. Naively one would have expected the partition function in the adjoint representation to be positive. What happens is that we have made an infinite subtraction due to the fact that the spectrum is unbounded below.

Now we would like to compare (3.18) with the matrix model answer. We are interested in computing the matrix model partition function in the adjoint representation. The matrix model answer depends on the scattering phase (3.14). We are then lead to a partition

[9] The computations in [21, 22] were off by a factor of two.

function for the adjoint of the form $Z_{adj} \sim Z_{sing} Z_W$, where Z_{sing} is the singlet partition function and Z_W is the partition function for the effective particle we are considering now.

$$
\begin{aligned}
Z_W &= \int d\epsilon \frac{\partial_\epsilon \delta(\epsilon)}{2\pi} e^{-2\pi R\epsilon} = -\mu^R \int_{-\infty}^{\infty} d\hat{\epsilon} \frac{1}{2} \hat{\epsilon} \left(\frac{1}{\tanh \pi\hat{\epsilon}} + 1\right) e^{-2\pi R\hat{\epsilon}} = -\frac{1}{4\sin^2 \pi R} \mu^R \\
&= -\frac{\mu^R}{4\pi^2} \Gamma(-R)^2 R^2 \Gamma(R)^2 \,.
\end{aligned} \tag{3.20}
$$

We see that we reproduce (3.18) up to factors that are analytic in R for positive R. The difference between (3.17) and (3.20) is reminiscent of the leg factors that appear when one compares scattering computations in string theory and the matrix model [3] but it is different in detail.[10] (3.20) is the correct normalization for the partition function of the adjoint since we constructed it explicitly as a trace over the Hilbert space. This normalization still has an ambiguity due to the precise constant in the subtraction of the energy. This means that that we can replace $\mu \to \mu c$ where c is an R independent numerical constant. Notice that the poles at integer values of R in (3.20) arise because when R is an integer we have a new divergence in the computation of the free energy since the integrand in (3.20) does not decay fast enough at infinity. This implies that a new subtraction becomes necessary. The coefficients of these divergent terms are integer powers of μ.

We now consider the problem from a slightly different point of view. Let us consider an FZZT brane extended in time and compute its cylinder diagram. This diagram contains a contribution from the exchange of closed strings that are wound in the time direction. We can pick out the contribution from strings that wind precisely once as follows. The cylinder diagram is (up to a positive normalization constant)

$$
Z_{cyl} \sim 2\pi R \sum_n \int_0^\infty dE \frac{\cos^2(\sigma\pi E)}{\sinh^2 \pi E} \frac{1}{E^2 + n^2 R^2} \,. \tag{3.21}
$$

The integer n is summing over the closed string exchanges with strings that are wound n times, so we will be interested in the term with $n = 1$. We disregard the pole at $E = 0$ and all other terms that are μ independent. We can rewrite the integral in (3.21) as an integral from $-\infty$ to ∞. Then we split the cosine into $e^{2i\sigma E}$ and its complex conjugate. Both terms give the same answer. For large σ it is convenient to shift the contour of the first to $E \to i\infty$. In this process we pick up poles at $E = im$ and $E = inR$. If $0 < R < 1$ the leading pole appears at $E = iR$ and its residue gives

$$
Z = -2\pi^2 e^{-2\sigma\pi R} \frac{1}{\sin^2 \pi R} \sim -\mu^R (\mu_B)^{-2R} \frac{1}{\sin^2 \pi R} \,. \tag{3.22}
$$

Note that also from this point of view we get a negative sign, in agreement with the worldsheet computation. Note that here, also, we started with (3.21) which is manifestly positive and, due to a subtraction, we ended up with a negative answer. The precise

[10]The leg factor that we need in this case to match the matrix model answer (3.20) to the string theory answer (3.18) is $\Gamma(R)^{-2}$ for each vertex operator, which looks different than the momentum leg factors, which are $\frac{\Gamma(-|p|)}{\Gamma(|p|)}$ (and we would set $p = R$).

agreement between (3.22) and (3.20) is simply a consistency check, since it is related to the fact that we assumed that the scattering phase in the matrix model agrees with the scattering phase in string theory.

We could take a scaling limit where we put N_f FZZT branes and then take

$$N_F, \mu_B \to \infty\,, \qquad \hat{\lambda} \equiv N_f \mu_B^{-R} = \text{fixed}\,. \tag{3.23}$$

(3.22) gives a finite source for the winding modes. This procedure would give a source only for the first winding mode. We should then identify $\hat{\lambda}$ with the term in the euclidean action considered in [2]

$$S = S_0 + \hat{\lambda}\,\mathrm{Tr}\left(e^{\int A}\right) + \hat{\lambda}\,\mathrm{Tr}\left(e^{\int A}\right)^{\dagger}\,. \tag{3.24}$$

Thinking of (3.24) as a limit of (3.23) has the advantage that we start from system with a clear lorentzian interpretation, at least on the string theory side, which in the limit (3.23) leads us to consider the partition function for the action (3.24) studied in [2]. So the two dimensional string theory in the presence of N_f FZZT branes might be a good starting point for thinking about the lorentzian version of the two dimensional black hole [1].

3.3 All orders result via T-duality

We expect the all orders partition function for the adjoint representation will be given the correlator of two strings with winding numbers one and minus one. We can compute this in the following indirect way. From the string theory point of view we can assume that T-duality is a good symmetry. On the T-dual side, this translates into a computation for momentum correlators. We can compute these using the matrix model fermions of the T-dual model, which are free since we are simply computing correlators of momentum operators. This is just a computation in the singlet sector and can be done using the general formulas in [31]. The matrix model fermions of the T-dual model should not be confused with the original matrix model fermions. T-duality is not an obvious symmetry of the matrix model. We find, see details in appendix D,

$$Z \sim Z_{sing}\left(\frac{\Gamma\left(\frac{1}{2}+i\mu R+\frac{R}{2}\right)\Gamma\left(\frac{1}{2}-i\mu R+\frac{R}{2}\right)}{\Gamma\left(\frac{1}{2}-i\mu R-\frac{R}{2}\right)\Gamma\left(\frac{1}{2}+i\mu R-\frac{R}{2}\right)}\right)^{1/2} \tag{3.25}$$

up to non perturbative terms in $\frac{1}{\mu}$, which is us much precision as we can hope for the bosonic string case. The correlator (3.25) is simply the momentum correlator in the T dual matrix model written in terms of the original variables. The result (3.25) differs from the result in [22] by the overall power (instead of $\frac{1}{2}$ in the exponent they had $\frac{1}{4}$[11]). On the T-dual matrix model we know that there are leg factors that need to be taken into account. Notice also that (3.18) differs from (3.20) by another leg factor. We do not understand the origin of these leg factors in a clear way. So assuming that they will work out as in the

[11] Equivalently, [22] obtain only the numerator in (3.25), which is equivalent to changing the overall power up to terms that are analytic in μ and non-perturbative in $1/\mu$.

usual singlet case we conclude that the all orders partition function in the singlet sector should be given by the the $1/\mu$ expansion of

$$
\begin{aligned}
Z &\sim -\frac{1}{4\sin^2 \pi R} Z_{\rm sing} \left(\frac{\Gamma(\frac{1}{2}+i\mu R+\frac{R}{2})\Gamma\left(\frac{1}{2}-i\mu R+\frac{R}{2}\right)}{\Gamma\left(\frac{1}{2}-i\mu R-\frac{R}{2}\right)\Gamma\left(\frac{1}{2}+i\mu R-\frac{R}{2}\right)} \right)^{1/2} \\
Z &\sim -Z_{\rm sing} \times \frac{\mu^R}{4\sin^2 \pi R}\left(1+\left(\frac{R}{24}-\frac{1}{24R}\right)\frac{1}{\mu^2}+\cdots\right)
\end{aligned}
\tag{3.26}
$$

where we introduced an R dependent factor in order to match the leading order answer to (3.20). These could come from leg factors which we do not fully understand.

3.4 More general representations

We saw that if we consider the adjoint representation we basically get one more particle. If we have a representation with n boxes and n antiboxes, then to leading order we will get a dilute gas of n particles which will have the same properties as the particle we had for the adjoint representation. The string theory dual will thus consist of n folded strings that come from the weak coupling region. The tip of these strings comes in, into the strong coupling region, and then goes back out to the weak coupling region. The details about the representation are related to symmetry properties of these particles, see [27] for further discussion.

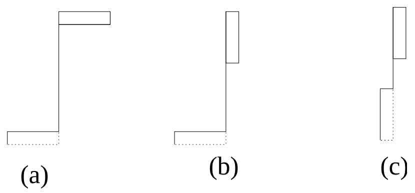

Figure 4: Young diagrams that give rise to SL(2) spin chains (a), SL(2) spin chains (c) and the supersymmetric Calogero model (b).

An interesting class of representations are those which are totally anti-symmetric in boxes and totally anti-symmetric in antiboxes. See figure 4(c). In this case we can map the problem into an SU(2) spin chain [27].

We can endow each matrix eigenvalue, which is a fermion, with an SU(2) spin and an interaction which is non-zero and goes as $1/(\lambda-\lambda')^2$ if the spins are in a singlet while it is zero if they are in the triplet of SU(2). Then we see the singlet sector corresponds to the case that all spins are up (we have maximal spin under global SU(2)). The adjoint corresponds to flipping one spin, and similarly a representation with n symmetric boxes and n symmetric anti-boxes corresponds to flipping n spins (most properly we should say that the total spin is $J^3 = N/2 - n/2$).

Another interesting set of representations are those completely symmetric in boxes and anti-boxes, see figure 4(a). In this case we get an SL(2) spin chain.

Finally we could consider representations which are anti-symmetric in boxes and symmetric in anti-boxes, see figure 4(b). In this case we also get some particles that move on the fermi sea, but they are fermionic. In fact, we can think of each fermion of the matrix model as endowed with an extra fermionic degree of freedom. The final theory is the supersymmetric Calogero-Moser model [32−34, 28, 35−40], with an inverted harmonic oscillator potential, with a total U(1) charge which is equal to $N-n$ where n is the number of boxes. The super-calogero model was proposed in [41] as a dual of string theory on AdS_2.

3.5 Two sided (or 0B) matrix model

In this subsection we consider a hermitian matrix model but we fill the two sides of the inverted harmonic oscillator potential. This is also called the 0B matrix model, since it is dual to type 0B string theory [42, 43]. Since we are choosing $\alpha' = \frac{1}{2}$ the matrix model potential is exactly the same as in the bosonic string theory case. So, up to equation (3.6) , we have the same discussion as in the bosonic string case. The new feature is that we need to consider both positive and negative values of λ. So we write $\lambda = \pm\sqrt{2\mu}\cosh\tau$ on these two sides. We also define the functions $h_{l,r}(\tau)$ which are defined on the left and right side of the inverted harmonic oscillator potential. Here we will assume that the Fermi level is below the barrier. As in the bosonic case we extend the range of the functions $h_{l,r}(-\tau) = -h_{l,r}(\tau)$ for $\tau \geq 0$. Then the eigenvalue equations become

$$Eh_r(\tau) = -\frac{1}{\pi}\int_{-\infty}^{\infty} d\tau' \left[\frac{h_r(\tau')}{4\sinh^2\frac{\tau-\tau'}{2}} + \frac{h_l(\tau')}{4\cosh^2\frac{\tau-\tau'}{2}}\right] + v(\tau)h_r(\tau) \tag{3.27}$$

with

$$v(\tau) \equiv \frac{2}{\pi}\left(\tau_{\max} - 1 - \frac{\tau}{\tanh\tau}\right) . \tag{3.28}$$

We have a similar equation with $r \leftrightarrow l$. Due to the Z_2 symmetry of the problem we can define h_+ and h_- where h_+ is such that $h_r(\tau) = h_l(\tau)$ and h_- is such that $h_r = -h_l$. The two kinetic terms that appear in (3.27) can be diagonalized to

$$\begin{aligned}
\hat{\epsilon}h &= K_\pm h + \hat{v}h \\
K_\pm &= \frac{k}{\tanh\pi k} \mp \frac{k}{\sinh\pi k} = k\left(\tanh\frac{\pi k}{2}\right)^{\pm 1} \\
\hat{v} &= -\frac{2}{\pi}\frac{\tau}{\tanh\tau} \\
\hat{\epsilon} &= E - \frac{2}{\pi}(\tau_c - 1) = E + \frac{2}{\pi}\phi_c + \frac{1}{\pi}\log\mu + \text{const} = \epsilon + \frac{1}{\pi}\log\mu\,.
\end{aligned} \tag{3.29}$$

where we have subtracted the divergent terms from the potential. We now see that the divergent energy is twice what we had in the bosonic case, which is what we expect after we identify $\sqrt{\mu}e^\tau \sim e^{-\phi}$ and use $\alpha' = \frac{1}{2}$. Note that the term involving $\log\mu$ in the definition of $\hat{\epsilon}$ has a different coefficient than in the bosonic string because of the factor of two in α'. In this units the Liouville cosmological constant term has the form $\mu\psi\bar{\psi}e^{2\phi}$. As in the bosonic case we were not able to solve the eigenvalue problem (3.29) directly. We have checked, however, that in the two asymptotic limits $\epsilon \to \pm\infty$ the leading and first subleading contributions computed from (3.29) match (2.21).

3.6 The vortex anti-vortex correlator for the superstring

In the type 0B superstring we have only one kind of winding string in the NS sector. As in the bosonic string we can compute the two point function using the three point function in [25].[12] This gives

[12]It is the one called $C_3(\alpha_i)$ in [25].

$$-\partial_\mu \langle e^{2(1-R)\phi} e^{2(1-R)\phi} \rangle_{\text{Liouv}} = i \langle \psi \bar{\psi} e^{2\phi} e^{2(1-R)\phi} e^{2(1-R)\phi} \rangle_{\text{Liouv}}$$
$$\sim i\mu^{2R-1} R^2 \frac{\Gamma(-R)}{\Gamma(R)} \langle 2pt \rangle = -\mu^{2R} R^2 \frac{\Gamma(-R)}{\Gamma(R)} \qquad (3.30)$$

where the last line is the answer in the full string theory computation, including the contribution due to the euclidean time direction.

We can now consider the computation of the euclidean partition function for the 0B matrix model in the adjoint representation. In this formula we will have to add over the two separate phase shifts

$$Z_W^{0B} = \int d\epsilon \frac{\partial_\epsilon (\delta_+(\epsilon) + \delta_-(\epsilon))}{2\pi} e^{-2\pi R \epsilon} = -\mu^{2R} \int_{-\infty}^{\infty} d\hat{\epsilon} \frac{1}{2} \hat{\epsilon} (\frac{1}{\tanh \pi \hat{\epsilon}} + 1) e^{-2\pi R \hat{\epsilon}} =$$
$$= -\frac{1}{4 \sin^2 \pi R} \mu^{2R} = -\frac{\mu^{2R}}{4\pi^2} R^2 \Gamma(-R)^2 \Gamma(R)^2 \,. \qquad (3.31)$$

We see that we get an answer that is essentially the same as the bosonic string (3.20), up to the replacement $\mu^R \to \mu^{2R}$. Again, this is the proper normalization.

We can also reproduce the result (3.31) by considering the cylinder diagram between two FZZT branes extended in time. This cylinder diagram comes purely from the exchange of NS-NS closed strings only because these FZZT branes do not have RR charge in 0B. This exchange has a form similar to (3.21) so the situation is very similar to what we encountered in the bosonic case.

4. Non-singlets in the complex matrix model and 0A superstrings

Two dimensional 0A superstring theory was conjectured to be dual to a $\mathrm{U}(N) \times \mathrm{U}(M)$ gauged matrix model, where the matrix M is in the $(N, \bar{M})$ representation of the gauge group [43]. In this section we consider the model in the presence of non-trivial representations. So we can choose representations $\mathcal{R}$, $\tilde{\mathcal{R}}$ of the two gauge groups and write

$$Z = \int \mathcal{D}m \mathcal{D}A \mathcal{D}\tilde{A} e^{i \int \mathrm{Tr}[(D_0 m)^\dagger (D_0 m) + V(m^\dagger m)]} \, \mathrm{Tr}_{\mathcal{R}}\, e^{i \int A} \, \mathrm{Tr}_{\tilde{\mathcal{R}}}\, e^{i \int \tilde{A}} \,. \qquad (4.1)$$

In a gauge where we set $A_0 = 0$ then we simply have a complex matrix model with the requirement that we keep only the states which transform in representations $\mathcal{R}^\dagger, \tilde{\mathcal{R}}^\dagger$ under $\mathrm{U}(N) \times \mathrm{U}(M)$. Now we can ask which representations are allowed. We obviously have that $\mathrm{U}(N) \times \mathrm{U}(M) \sim \mathrm{U}(1) \times \mathrm{SU}(N) \times \tilde{\mathrm{U}}(1) \times \tilde{\mathrm{SU}}(M)$. Clearly the representations cannot transform under the overall $\mathrm{U}(1)$, so $Q = -\tilde{Q}$. In this paper we will consider representations whose charges, including $Q, \tilde{Q}$, remain fixed as $N \to \infty$. We will also keep $M - N$ finite.[13] In this case, we have that

$$\tilde{Q} = Q_N = -Q = -\tilde{Q}_M \qquad (4.2)$$

where Q_N and $\tilde{Q}_M$ are the N-alities of the representations of $\mathrm{SU}(N)$ and $\mathrm{SU}(M)$ respectively. If we keep the representations "fixed" when $N \to \infty$ we can think of the N-alities as well defined integers.

[13] It is also interesting to consider the case when $Q = -\tilde{Q} \sim \hat{q} N$, which correspond to adding a certain RR flux in the string theory. We consider this case in a separate publication [44].

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Using methods similar to the ones used to derive (3.2) we can derive the reduced hamiltonian for a complex matrix acting on wavefunctions that transform under representations $\mathcal{R}$, $\tilde{\mathcal{R}}$

$$
\begin{aligned}
H = & \Bigg[\sum_{i=1}^{N} -\frac{1}{2}\frac{\partial^2}{\partial \rho_i} - \frac{1}{2}\rho_i^2 + \frac{1}{2}\frac{(\Pi_i^i)^2 + (N-M)^2 - \frac{1}{4}}{\rho_i^2} + \\
& + 2\sum_{i<j\le N} \frac{(\rho_i^2+\rho_j^2)(\Pi_j{}^i\Pi_i{}^j + \tilde{\Pi}_j{}^i\tilde{\Pi}_i{}^j) + 2\rho_i\rho_j(\Pi_j{}^i\tilde{\Pi}_i{}^j + \tilde{\Pi}_j{}^i\Pi_i{}^j)}{(\rho_i^2-\rho_j^2)^2} + \\
& + \sum_{i=1}^{N}\frac{1}{\rho_i^2}\sum_{j>N}(\tilde{\Pi}_j{}^i\tilde{\Pi}_i{}^j + \tilde{\Pi}_i{}^j\tilde{\Pi}_j{}^i)\Bigg] P_0
\end{aligned} \tag{4.3}
$$

where P_0 is a projector on the states obeying

$$
\begin{aligned}
\Pi_i^i + \tilde{\Pi}_i^i &= 0 \qquad \text{(no sum)} \\
\tilde{\Pi}_l{}^k &= 0\,, \qquad l,k > N
\end{aligned} \tag{4.4}
$$

where Π_{ij} are the U(N) generators and $\tilde{\Pi}_{ij}$ are the U(M) generators. The last line implies that under the decomposition SU(N) $\times$ SU$(M-N)$ $\times$ U(1) $\subset$ SU(M) we consider only states that are trivial under SU$(M-N)$. I have assumed that $M \ge N$.

4.1 Two simple examples of non-trivial representations

Consider $N = M$ and pick a state in the adjoint of SU(N) and the trivial representation in $\tilde{\mathrm{U}}(M)$. So $\tilde{\Pi}_i^j = 0$. Then $\Pi_{ii} = 0$ (no sum) and the hamiltonian becomes

$$
H = \sum_i \left(-\frac{1}{2}\partial_{\rho_i}^2 - \frac{1}{2}\rho_i^2 + \frac{1}{2}\frac{(-\frac{1}{4})}{\rho_i^2} \right) + 2\sum_{i<j}\frac{\rho_i^2+\rho_j^2}{(\rho_i^2-\rho_j^2)^2}\Pi_i^j\Pi_j^i\,. \tag{4.5}
$$

The term involving the one quarter is subleading at large μ, so I neglect it.[14] As in the hermitian matrix model we can now write the eigenvalue problem as

$$
Ew(\lambda) = 2\int d\lambda' \rho(\lambda')\frac{\lambda^2+\lambda'^2}{(\lambda^2-\lambda'^2)^2}(w(\lambda)-w(\lambda'))\,. \tag{4.6}
$$

We then use a formula similar to (3.6) to find a kinetic term and a potential which turns out to be exactly the same as what we had for the 0B model. The crucial aspect is that we can write

$$
2\frac{\lambda^2+\lambda'^2}{(\lambda^2-\lambda'^2)^2} = \frac{1}{(\lambda-\lambda')^2} + \frac{1}{(\lambda+\lambda')^2}\,. \tag{4.7}
$$

So the problem is basically the same as the problem we had for the 0B case for the so called even functions. We expect then that the scattering phase is given by δ_+ in (2.19). Obviously, we would get the same answer if we considered a representation which is trivial in SU(N) and is the adjoint of $\tilde{\mathrm{SU}}(N)$.

[14] It is not difficult to include it [43].

Another interesting representation is a fundamental of the first and an anti-fundamental of the second gauge group. Again we can label the states in the representation by W_i^j. Then the constraint on the diagonal part forces W to be diagonal. We then find an eigenvalue equation of the form

$$Ew_i = \frac{1}{2}\frac{1}{\lambda_i^2}w_i + w_i \sum_j \left(\frac{1}{(\lambda_i - \lambda_j)^2} + \frac{1}{(\lambda_i + \lambda_j)^2} \right) - 2\sum_j \left(\frac{1}{(\lambda_i - \lambda_j)^2} - \frac{1}{(\lambda_i + \lambda_j)^2} \right) w_j \tag{4.8}$$

where we used that

$$\frac{4\lambda\lambda'}{(\lambda^2 - \lambda'^2)^2} = \frac{1}{(\lambda - \lambda')^2} - \frac{1}{(\lambda + \lambda')^2}\,. \tag{4.9}$$

So we get a problem which is similar to the 0B problem for odd wavefunctions. The only difference is a potential term which goes like $1/\lambda^2$. This term is subleading in the $1/\mu$ expansion, so we ignore it. In conclusion, we expect that the scattering phase is given by δ_- in (2.19).

4.2 Partition functions and cylinder diagram for FZZT branes

Using the phase shifts, we can easily compute the partition functions for the $\pm$ cases. We obtain

$$\begin{aligned} Z^+ &\sim \int d\epsilon \frac{\partial_\epsilon \delta_+(\epsilon)}{2\pi} e^{-2\pi\epsilon} = -\mu^{2R} \frac{\cos 2\pi R}{2\sin^2 2\pi R} \\ Z^- &\sim \int d\epsilon \frac{\partial_\epsilon \delta_-(\epsilon)}{2\pi} e^{-2\pi\epsilon} = -\mu^{2R} \frac{1}{2\sin^2 2\pi R}\,. \end{aligned} \tag{4.10}$$

Of course, $Z^+ + Z^-$ gives us again (3.31).

Now let us consider the vortex anti-vortex correlator in 0A theory. Winding modes can be in the NS-NS or RR sector. The NS-NS correlator was computed in (3.30). In order to compute the RR correlator we consider the derivative with respect to μ of the two point function. We then have to compute a three point function of one NS and two RR vertex operators. We put the NS operator in the -1 picture and the RR operators in the $-1/2$ picture. So we will need the super Liouville correlator in [25].[15] We obtain

$$\begin{aligned} \partial_\mu \langle 2pt \rangle &\sim -R\mu^{2R-1} \frac{\Gamma\left(\frac{1}{2} - R\right)^2}{\Gamma\left(\frac{1}{2} + R\right)^2} \\ \langle 2pt \rangle &= -\mu^{2R} \frac{\Gamma\left(\frac{1}{2} - R\right)^2}{\Gamma\left(\frac{1}{2} + R\right)^2}\,. \end{aligned} \tag{4.11}$$

[15] It is the one called $\tilde{C}_3$ in [25].

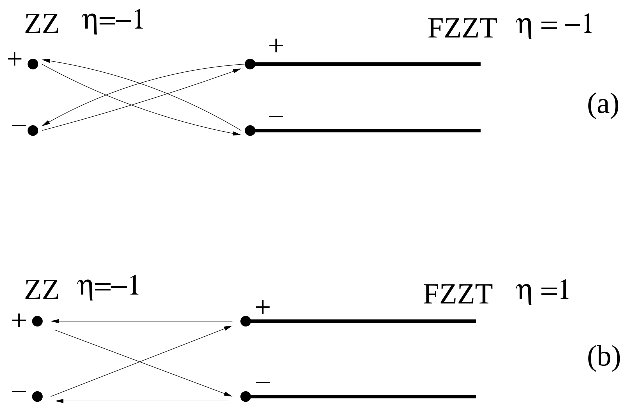

Figure 5: Spectrum of open strings between ZZ branes and FZZT branes extended in time in 0A. In (a) we see the spectrum between ZZ branes and FZZT branes with the same η. In this case the ZZ and FZZT branes are charged under the same gauge field. In (b) we consider an FZZT brane with opposite η. Here the ZZ and FZZT branes are charged under a different gauge field. In both cases only the region where the FZZT branes dissolve are charged. This region is represented here as a dot, but it is not sharply localized in target space.

We can also compute the cylinder diagrams for FZZT branes in the 0A theory. In the 0A case the FZZT branes carry RR charge. So the cylinder diagrams are a bit more interesting than in the 0B case. The NS contribution is similar to that of the 0B theory, and the bosonic theory. The RR contribution depends on the sign of η. For example, for $\eta = -1$ we have

$$\begin{aligned} Z_{cyl}^{\eta=-1} &= (2\pi R)\sum_n \int dE \left(\frac{\cos^2 2\pi E\sigma}{\sinh^2 \pi E} + \frac{\cos^2 2\pi E\sigma}{\cosh^2 \pi E} \right) \frac{1}{E^2 + n^2R^2} \\ Z_{cyl}^{\eta=-1}\Big|_{n=1} &\sim -\left(\frac{1}{\sin^2 \pi R} - \frac{1}{\cos^2 \pi R} \right) \mu^{2R} \mu_B^{-4R} \end{aligned} \tag{4.12}$$

where in the second line we have extracted the contribution from the first pole at $E = iR$.

Similarly for the $\eta = 1$ brane we get

$$\begin{aligned} Z_{cyl}^{\eta=1} &= (2\pi R)\sum_n \int dE \left(\frac{\cos^2 2\pi E\sigma}{\sinh^2 \pi E} - \frac{\sin^2 2\pi E\sigma}{\cosh^2 \pi E} \right) \frac{1}{E^2 + n^2R^2} \\ Z_{cyl}^{\eta=1}\Big|_{n=1} &\sim -\left(\frac{1}{\sin^2 \pi R} + \frac{1}{\cos^2 \pi R} \right) \mu^{2R} \mu_B^{-4R} . \end{aligned} \tag{4.13}$$

In the computation of the cylinder diagrams we did not keep track of the overall (positive) numerical normalization. Up to this constant they agree with (4.10) , after we identify $Z^+ \sim Z_{cyl}^{\eta=-1}|_{n=1}$ and $Z^- \sim Z_{cyl}^{\eta=-1}|_{n=1}$. The second term in (4.12) (4.13) comes from the RR exchange and it has a form that is similar to (4.11).

What these results are saying is that the open strings that stretch between the FZZT and ZZ branes should have the following properties. See figure 5. Let us set $\mu > 0$. We have ZZ branes with two possible charges. Let us consider the open strings stretching between the ZZ branes and the FZZT branes with the same η as the ZZ branes. In this case we expect that the open strings go between the FZZT brane and the ZZ brane of a given charge, say plus. The charge they go to is related to the charge living at the tip of the FZZT brane. Now let us consider the open strings between the ZZ and the FZZT brane with the opposite η than the ZZ brane. We expect that the open strings end on the charge plus or minus ZZ brane depending on their orientation. Notice that in this case the ZZ and FZZT branes are charged under different spacetime gauge fields.

Could we see this properties directly from the string theory?. Let us think about Liouville theory as a rational conformal field theory. Then according to the Cardy construction the various branes are labeled by the primary fields. The two ZZ branes are labeled by $1, \psi$, depending on their RR charge. The FZZT branes with the same η as the ZZ branes will be labeled by $e^{(1+i\sigma)\phi}$ and $\psi e^{(1+i\sigma)\phi}$. The open string spectrum is given by taking the OPE of these fields. We see that this produces states that have the form $e^{(1+i\sigma)\phi}$ or $\psi e^{(1+i\sigma)\phi}$. For generic σ only one of these two states is a superconformal primary. After restricting to superconformal primary states we see we have the open string spectrum with the properties we sketched above. FZZT branes with the opposite η are labeled by Ramond states of the form $R_{\pm}e^{(1+i\sigma)\phi}$. In this case the OPE with the states $1, \psi$ will produce states of the form $R_{\pm}e^{(1+i\sigma)\phi}$ and $R_{\mp}e^{(1+i\sigma)\phi}$. Again only half of these states are superconformal primaries annihilated by the supercharge G_0. When we include X the surviving states will depend on the orientation of the string. This produces a string spectrum which is consistent with the one we needed on the basis of the matrix model answers.

5. Discussion

We studied the matrix model in its non-singlet sector. After subtracting a divergence we saw that the matrix model hamiltonian is unbounded below in this sector. Nevertheless, there are very natural lorentzian computations which involve scattering amplitudes. These amplitudes involve long strings in the target space. These are folded strings that stretch from the weak coupling region. The tip of the string moves under the force provided by the string tension. We considered in detail the case of the adjoint representation (and also bifundamental, for 0A). In this case we have a single long string. More complicated representations, whose Young diagrams involve n boxes and n anti-boxes, correspond to states with n folded long strings. The string theory results match the matrix model results, in the region where we could compute them explicitly.

This picture helps us understand the physical meaning of euclidean computations for winding correlators. In the matrix model these are often computed by appealing to T-duality and then computing momentum correlators in the T-dual free fermion picture. This helps us understand, for example, the sign of the two point function for winding correlators.

Probably the picture of non-singlets proposed in this paper will be helpful for finding a matrix model picture for the lorentzian black hole. The action proposed in [2] for the euclidean black hole does not have an obvious *local* lorentzian continuation. It is likely that by taking a scaling limit with a large number of FZZT branes (3.23) , we could find a satisfactory lorentzian picture for the black hole. This point of view would suggest that we should look for it as an excited long lived state in the non-singlet sector provided by the FZZT branes in the limit (3.23).

Note that the mode that changes the value of the dilaton at the tip is not normalizable in $SU(2)_k/U(1)$ for $k < 3$ (see section 7.3 of [45]). This implies that the thermodynamics of the cigar is qualitatively different than in the large k situations, which was analyzed in [46, 47].[16] This fact explains that despite appearances we can actually change the radius of the cigar in two dimensional string theory. It also explains why there is no sign of the black hole in the singlet sector thermal partition function [48].

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A. Derivation of the scattering amplitudes

A.1 Bosonic string

We start by taking the $b \to 1$ limit of the formulas in [20]. The two point function for the scattering of two open strings with two ends on the same D-brane is

$$\begin{aligned} \langle e^{(1+i\alpha)\phi} e^{(1+i\alpha)\phi} \rangle &= \mu^{-i\alpha} \frac{\Gamma(2i\alpha)G(1-2i\alpha)G(1+i\alpha+is)G(1+i\alpha)^2}{\Gamma(-2i\alpha)G(1+2i\alpha)G(1-i\alpha-is)G(1-i\alpha)^2} \frac{G(1+i\alpha-is)}{G(1-i\alpha+is)} \\ \cosh \pi s &= \frac{\mu_B}{\sqrt{\mu}} \qquad\qquad (A.1) \\ ig(\alpha) \equiv \log \frac{G(1+i\alpha)}{G(1-i\alpha)} &= i\frac{1}{2}\int_0^\infty \frac{dt}{t}\left[\frac{\sin 2t\alpha}{\sinh^2 t} - \frac{2\alpha}{t}\right] = -i\pi \int_0^\alpha d\alpha' \frac{\alpha'}{\tanh \pi\alpha'} \,. \end{aligned}$$

In the $b \to 1$ limit we also need to scale the bulk and boundary cosmological constants to get a finite answer. In (A.2) we present the answer in terms of the rescaled variables. Note that the function $g(\alpha)$ has the behavior

$$g(\alpha) \sim \mp\frac{\pi}{2}\alpha^2 \qquad \text{for } \alpha \to \pm\infty \qquad (A.2)$$

up to exponential corrections. We now take the scaling limit

$$s \to \infty\,, \qquad \alpha \to \infty\,, \qquad \epsilon \equiv \alpha - \frac{1}{\pi}\log 2\mu_B = \alpha - s - \frac{1}{2\pi}\log\mu = \text{finite} \qquad (A.3)$$

[16] In particular, section 6 of [2] does not seem to apply to the two dimensional black hole.

We see that the last factor in the first line in (A.2) is finite. There is also a finite contribution from other terms. This arises because we want to write every α as $\alpha = s + \hat{\epsilon}$ and then take the limit. Using (A.2) we find that the scattering amplitude is

$$\begin{aligned}
e^{i\delta_{FZZ}} &= e^{i\delta(\epsilon)} e^{i\delta_{\rm def}(\epsilon)} e^{i\delta_{\rm div}(\epsilon)} \\
\delta &= g\left(\epsilon + \frac{1}{2\pi}\log\mu\right) - \frac{\pi}{2}\left(\epsilon + \frac{1}{2\pi}\log\mu\right)^2 \\
\delta_{\rm def} &= \frac{1}{\pi}(\phi_c + \pi\epsilon)^2 \\
\delta_{\rm div} &= -2\phi_c\left(\epsilon + \frac{1}{\pi}\phi_c\right) + \frac{4}{\pi}\phi_c\left(\frac{\log 2\phi_c}{\pi} - 1\right) + \frac{4\epsilon\log 2\phi_c}{\pi}, \qquad \phi_c \equiv \log(2\mu_B)
\end{aligned} \tag{A.4}$$

where the phase δ_{def} is a phase that appears in the definition of the scattering phase in (2.7). The first term in the divergent piece can be interpreted as coming from the propagation from infinity to ϕ_c. The rest of the terms in the divergent amplitude seem related to the details of the regularization procedure through the FZZT brane. In other words, they are probably related to the particular way that the ends of the string interact with the boundary Liouville potential. We can forget about the divergent piece. Notice that the divergent pieces are independent of μ. This is a consistency check since all these divergent pieces are related to our particular regularization of the amplitude using FZZT branes and these branes are dissolving far away from the region where the Liouville cosmological constant is important. After defining

$$\delta(\epsilon) \equiv g(\hat{\epsilon}) - \frac{\pi}{2}\hat{\epsilon}^2 \tag{A.5}$$

we recover (2.15).

In the limit when $\epsilon \to -\infty$, then $\delta \to 0$ in agreement with (2.8). When $\epsilon \to +\infty$ we get

$$e^{i\delta} e^{i\pi\epsilon^2} \sim \mu^{-i(\epsilon + \frac{1}{4\pi}\log\mu)}\,. \tag{A.6}$$

The energy dependence is the precisely the one we get for the scattering of a massless "tachyon" from the Liouville potential. We have included a factor of $e^{i\pi\epsilon^2}$ which comes from the difference between the definition of the scattering amplitude as in (2.7) versus the more standard definition which would lack such factor. This more standard definition is the one used to study the scattering of massless particles. We have chosen the apparently more complicated definition (2.7) because with this definition the scattering phase depends only on $\epsilon + \frac{1}{2\pi}\log\mu$. Had we chosen a more standard definition via

$$\psi \sim e^{-i\frac{1}{2\pi}\phi^2 + i\epsilon\phi} + e^{i\delta_{usual}} e^{i\frac{1}{2\pi}\phi^2 - i\epsilon\phi} \tag{A.7}$$

then we would find that

$$\delta_{usual} = \delta + \pi\epsilon^2 + \pi \tag{A.8}$$

where δ is defined in (2.7). Notice that it is ϵ that appears here and not $\hat{\epsilon} = \epsilon + \frac{1}{2\pi}\log\mu$. While δ depends only on $\hat{\epsilon}$, δ_{usual} in (A.8) depends on both $\hat{\epsilon}$, ϵ, but the dependence on the latter is very simple.

A.2 Type 0 superstring

In order to compute the scattering amplitudes for the superstring we start with the exact formulas of for the two point function in [25]. Again we just present the $b \to 1$ limit of the formulas in [25]. Let us consider first the scattering of NS open strings which stretch between FZZT branes with the same η. The result for the scattering of two such open strings depends on η. Let us consider, for example, the amplitude for the scattering of open strings on the $\eta = -1$ FZZT brane, in conventions where the ZZ brane has $\eta = -1$. This gives[17]

$$\begin{aligned}
\langle e^{(1+i\alpha)\phi} e^{(1+i\alpha)\phi} \rangle_{\eta=-1} &= \mu^{-i\alpha} \frac{G(1-i\alpha)^2 \Gamma(i\alpha)}{G(1+i\alpha)^2 \Gamma(-i\alpha)} \frac{G_{NS}(1+i\alpha+2is)}{G_{NS}(1-i\alpha-2is)} \frac{G_{NS}^2(1+i\alpha)}{G_{NS}^2(1-i\alpha)} \times \\
&\quad \times \frac{G_{NS}(1+i\alpha-2is)}{G_{NS}(1-i\alpha+2is)} \\
G_{NS}(x) &= G\left(\frac{x}{2}\right) G\left(\frac{x+1}{2}\right) \\
\cosh \pi s &= \frac{\mu_B}{\sqrt{\mu}} \\
i g_+(\alpha) &\equiv \log \frac{G_{NS}(1+i\alpha)}{G_{NS}(1-i\alpha)} = ig\left(\frac{\alpha}{2} + \frac{i}{2}\right) + ig\left(\frac{\alpha}{2} - \frac{i}{2}\right) \\
&= -i2 \int_0^{\frac{\alpha}{2}} d\alpha' \alpha' \pi \tanh \alpha' \pi \qquad \text{(A.9)}
\end{aligned}$$

where $G(x)$, and $g(\alpha)$ are the same functions as in (A.2) . In these formulas α is equal to the energy of the open string. Using the asymptotic behavior of $g_{NS}(\alpha)$ as $\alpha \to \pm\infty$ we can take the scaling limit

$$s \to \infty\,, \qquad \alpha \to \infty\,, \qquad \epsilon \equiv \alpha - \frac{2}{\pi} \log 2\mu_B = \left(\alpha - 2s - \frac{2}{2\pi} \log \mu\right) = \text{finite}\,. \qquad \text{(A.10)}$$

After throwing away divergent pieces and pulling out a piece of the required form to define the phase as in (2.7) we find

$$\begin{aligned}
\delta_+ &= -2 \int_{-\infty}^{\hat{\epsilon}/2} d\alpha' \alpha' \pi (\tanh \pi\alpha' + 1) \\
\hat{\epsilon} &\equiv \epsilon + \frac{1}{\pi} \log \mu \qquad \text{(A.11)}
\end{aligned}$$

Note that $\hat{\epsilon}$ for the superstring is defined differently than in the bosonic string, due to the factor of two in the value of α'. Note that (A.11) is such that it goes to zero as $\alpha \to 0$, in accordance with (2.8).

We now turn our attention to the case of an open string that lives on an FZZT brane with $\eta = 1$. In this case we obtain

$$\langle e^{(1+i\alpha)\phi} e^{(1+i\alpha)\phi} \rangle_{\eta=1} = \mu^{-i\alpha} \frac{G(1-i\alpha)^2 \Gamma(i\alpha)}{G(1+i\alpha)^2 \Gamma(-i\alpha)} \frac{G_R(1+i\alpha+2is)}{G_R(1-i\alpha-2is)} \frac{G_{NS}^2(1+i\alpha)}{G_{NS}^2(1-i\alpha)} \times$$

[17] Notice that $2\phi_{here} = \phi_{Fukuda-Hosomichi}$ in [25].

$$\begin{aligned}
&\times\frac{G_R(1+i\alpha-2is)}{G_R(1-i\alpha+2is)}\\
G_R(x) &= G\left(\frac{x}{2}+\frac{1}{2}\right)^2 && \text{(A.12)}\\
\cosh\pi s &= \frac{\mu_B}{\sqrt{\mu}} ig_-(\alpha)\equiv \log\frac{G_R(1+i\alpha)}{G_R(1-i\alpha)} = i2g\left(\frac{\alpha}{2}\right) = -i2\int_0^{\frac{\alpha}{2}} d\alpha'\frac{\alpha'\pi}{\tanh\alpha'\pi}
\end{aligned}$$

where G_{NS} is the same as the one defined above. Notice that, despite the R index, we are scattering an NS open string. We are just using the notation in [25]. Taking the same limit as in (A.10) we find the phase

$$\delta_- = -2\int_{-\infty}^{\frac{\hat{\epsilon}}{2}} d\epsilon'\epsilon'\pi\left(\frac{1}{\tanh\epsilon'\pi}+1\right). \tag{A.13}$$

B. Computation of matrix model reflection amplitudes

B.1 Derivation of (3.12)

We can rewrite (3.6) as

$$\begin{aligned}
Eh(\tau) &= -\frac{1}{\pi}\int_0^\infty d\tau'\frac{\sinh\tau\sinh\tau'}{(\cosh\tau-\cosh\tau')}h(\tau')+v(\tau)h(\tau)\\
v(\tau) &= \frac{1}{\pi}\int_0^\infty d\tau'\frac{\sinh^2\tau'}{(\cosh\tau-\cosh\tau')^2}\\
1 &= \frac{1}{\pi}\int_0^\infty d\tau h(\tau)^2\,.
\end{aligned}\tag{B.1}$$

It is possible to rewrite the kinetic term using

$$\frac{1}{\pi}\frac{\sinh\tau\sinh\tau'}{(\cosh\tau-\cosh\tau')^2} = \frac{1}{\pi}\frac{1}{4\sinh^2\frac{\tau-\tau'}{2}} - \frac{1}{\pi}\frac{1}{4\sinh^2\frac{\tau+\tau'}{2}} \tag{B.2}$$

Similarly the potential can be written as

$$v(\tau) = \frac{1}{\pi}\left(\tau_{max}-1-\frac{\tau}{\tanh\tau}\right). \tag{B.3}$$

Since the eigenvalue distribution goes to zero at $\tau=0$ it is reasonable to assume that $h(\tau=0)=0$. Note that $h(\tau)$ is initially defined only for $\tau>0$. It is reasonable to extend the definition of $h(\tau)$ on the whole real axis by defining it as an odd function of τ. Namely, $h(\tau)=-h(-\tau)$ for $\tau<0$. Then we can rewrite the kinetic term as an integral in τ' from minus to plus infinity. In other words we find that

$$-\frac{1}{\pi}\int_0^\infty d\tau'\frac{\sinh\tau\sinh\tau'}{(\cosh\tau-\cosh\tau')}h(\tau') = -\frac{1}{\pi}\int_{-\infty}^\infty d\tau'\frac{h(\tau')}{4\sinh^2\frac{\tau-\tau'}{2}}\,. \tag{B.4}$$

We are then left with (3.12) (3.13).

B.2 Some scattering amplitudes from the matrix model

Let us discuss solutions of (3.12). Let us first assume that $\hat{\epsilon} \ll 0$. In this case we can work in Fourier space and approximate the potential as $\frac{\tau}{\tanh \tau} \sim \tau = i\partial_k$. Then we get the equation

$$\hat{\epsilon} h(k) = \left(\frac{k}{\tanh \pi k} - i\frac{1}{\pi}\partial_k \right) h(k) \tag{B.5}$$

which is solved by

$$\begin{aligned} h(k) &= \mathrm{Exp}\left(i\pi\hat{\epsilon}k - i\pi \int_0^k dk' \frac{k'}{\tanh \pi k'} \right) \\ h(k) &\longrightarrow \mathrm{Exp}\left(i\pi\hat{\epsilon}k \mp \frac{\pi}{2}k^2 - \frac{i\pi}{12} \right) \qquad \text{as } k \to \pm\infty \end{aligned} \tag{B.6}$$

Its Fourier transform for large τ can be done by saddle point approximation and we get

$$h(\tau) \sim e^{-i\frac{1}{2\pi}(\tau+\pi\hat{\epsilon})^2} - e^{i\frac{1}{2\pi}(\tau+\pi\hat{\epsilon})^2}, \qquad \tau \gg 1 \tag{B.7}$$

where we ignored an overall phase. We then conclude that the scattering phase is zero. In general, we are defining the phase through

$$h(\tau) \sim e^{-i\frac{1}{2\pi}(\tau+\pi\hat{\epsilon})^2} - e^{i\delta} e^{i\frac{1}{2\pi}(\tau+\pi\hat{\epsilon})^2}, \qquad \tau \gg 1\,. \tag{B.8}$$

We can now compute the first correction by expanding the potential term $\frac{\tau}{\tanh \tau} \sim \tau + 2\tau e^{-2\tau}$ and writing $h = h_0(1+v_1)$ where h_0 is the leading order solution (B.6). We then get the equation

$$\partial_k v_1 = -2h_0^{-1}(k)\partial_k[e^{-2i\partial_k} h_0(k)] = 2(-i\pi\hat{\epsilon} + i\frac{\pi(k-2i)}{\tanh \pi k})e^{-i\pi\int_k^{k-2i} \frac{k'}{\tanh \pi k'}} e^{2\pi\hat{\epsilon}}\,. \tag{B.9}$$

We can evaluate the integral in the exponent of (B.9) to obtain

$$\begin{aligned} v_1(\infty) - v_1(-\infty) &= -2ie^{2\pi\hat{\epsilon}} \int_{-\infty}^{\infty} \left(\pi\hat{\epsilon} - \pi\frac{k-2i}{\tanh \pi k} \right) \frac{1}{4\sinh^2 \pi k} \\ &= i(\hat{\epsilon} - \frac{1}{2\pi})e^{2\pi\hat{\epsilon}} \end{aligned} \tag{B.10}$$

where we integrated by shifting the contour away from the singularity at $k = 0$. We see that we reproduce (2.16).

Let us now study the opposite regime, namely $\hat{\epsilon} \gg 0$. We can approximate the kinetic term as $K(k) = k$. Then the equation and the solution becomes

$$\begin{aligned} 0 &= \left(\hat{\epsilon} + i\partial_\tau + \frac{\tau}{\pi \tanh \tau} \right) h(\tau) \\ h(\tau) &= e^{i\hat{\epsilon}\tau + i\frac{1}{\pi}\int_0^\tau \frac{\tau'}{\tanh \tau'} d\tau'} \\ h(\tau) &\sim e^{i\tau\hat{\epsilon} + i\frac{\tau^2}{2\pi} + i\frac{\pi}{12}}, \qquad \tau \gg 0 \\ h(\tau) &\sim e^{i\tau\hat{\epsilon} - i\frac{\tau^2}{2\pi} + i\frac{\pi}{12}}, \qquad \tau \ll 0 \end{aligned} \tag{B.11}$$

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In order to solve the boundary conditions at the origin we still need to impose that $h(\tau) = h(-\tau)$. This can be done by taking $h \to h(\tau) - h(-\tau)$ with h as defined above. Comparing with (B.8) we see that the leading order phase is $\delta = -\pi\hat{\epsilon}^2$ in agreement with the leading order term in (2.17). Now we can do a first order expansion. We expand the kinetic term as $E(k) = k + 2ke^{-2\pi k}$. The exponential is an operator performing a translation $\tau \to \tau + 2\pi i$. As before we write $h = h_0(1 + v_1)$ and we get

$$i\partial_\tau v_1 = -2i\frac{\partial_\tau h_0(\tau + 2\pi i)}{h_0(\tau)} = 2\left(\hat{\epsilon} + \frac{\tau + 2\pi i}{\pi \tanh \tau}\right) e^{i\int_\tau^{\tau+2\pi i} \frac{\tau'}{\pi \tanh \tau'} d\tau'} e^{-2\pi\hat{\epsilon}} . \tag{B.12}$$

We can evaluate the integral in the exponent of (B.12) to obtain

$$v_1(\infty) - v_1(-\infty) = -ie^{-2\pi\hat{\epsilon}} \int_{-\infty}^{\infty} \left(\hat{\epsilon} + \frac{\tau + 2\pi i}{\pi \tanh \tau}\right) \frac{1}{2\sinh^2 \tau} = i\left(\hat{\epsilon} + \frac{1}{2\pi}\right) e^{-2\pi\hat{\epsilon}} \tag{B.13}$$

where again we have shifted the contour to $\tau \to \tau + i\pi$. We see then that we reproduce the subleading term in (2.16).

For the two sided matrix model we can also compute the leading terms and the first correction. The whole computation is very similar to the one we did above, except for factors of two. When we compute the first subleading correction starting from high energies $\hat{\epsilon} \gg 0$, then it is easy to see that the results for $\delta_\pm$ differ by just a sign, as we expect from the high energy expansion of (2.19) which is

$$\begin{aligned} \delta_- &\sim -\frac{\pi}{2}\hat{\epsilon}^2 + \left(\hat{\epsilon} + \frac{1}{\pi}\right) e^{-\pi\hat{\epsilon}} , \qquad \hat{\epsilon} \gg 0 \\ \delta_+ &\sim -\frac{\pi}{2}\hat{\epsilon}^2 - \left(\hat{\epsilon} + \frac{1}{\pi}\right) e^{-\pi\hat{\epsilon}} , \qquad \hat{\epsilon} \gg 0 . \end{aligned} \tag{B.14}$$

The difference in sign comes from the difference in sign when we expand the kinetic terms in (3.29) , $K_\pm \sim k \mp 2ke^{-\pi k}$.

For very low energies $\epsilon \ll 0$ we can do a similar expansion. Clearly for δ_- we will get a result similar to the one sided model, in agreement with (2.21). For δ_+ the leading correction will have the opposite sign which comes, basically, from the fact that we will have a $(\cosh \pi k/2)^{-2}$ instead of $(\sinh \pi k/2)^{-2}$ in an expression similar to (B.10). Going through all the details one indeed finds the opposite sign, in agreement with (2.21).

In conclusion, we have checked explicitly that the leading and first subleading terms computed in the matrix model agree with the string theory answers. This holds both for the bosonic string and the superstring cases. It is interesting that the same functions are appearing in the potential, the kinetic terms and the phase. This suggests that there must be a simpler way to think about problem this that would allow us to get the answer more quickly.

C. The FZZ conjecture

We start from the $N = 2$ version of the FZZ conjecture, which was considered in [49] , see also [50]. This version of the conjecture is defined for any value of k, regardless of whether the perturbation away from the linear dilaton region is or is not normalizable.

We now observe that the cigar theory with worldsheet supersymmetry is the same as a bosonic cigar plus two free fermions. So there should be some free fields also on the $\mathcal{N}=2$ sine Liouville side. It is not difficult to identify the free field, it is just the U(1) current of the $\mathcal{N}=2$ superconformal algebra. After quotienting by this current we get the bosonic $\mathcal{N}=2$ sine Liouville. So we conclude that the bosonic version of the SineLiouville/cigar duality derives from the $\mathcal{N}=2$ version studied in [49].

Let us see this a bit more explicitly.[18] Let us denote by k_s the level of the $\mathcal{N}=2$ cigar theory. Let us also define $k_b = k_s + 2$, which is the level of the associated bosonic cigar theory. Then the $N=2$ Liouville theory has the interaction terms

$$\mu e^{i\sqrt{2}\sigma} e^{\sqrt{k_s}\phi + i\sqrt{k_s}(\varphi_L - \varphi_R)} \tag{C.1}$$

where σ accounts for the bosonized fermions. We now consider the expression for the $\mathcal{N}=2\,\mathrm{U}(1)$ current and we define a new scalar field x, which is orthogonal to it

$$\begin{aligned} j &\sim \sqrt{2}\left(\partial\sigma_L - \frac{\sqrt{2}}{\sqrt{k_s}}\partial\varphi_L\right) \\ \sqrt{k_b}x_L &\equiv \sqrt{2}\sigma_L + \sqrt{k_s}\varphi_L \end{aligned} \tag{C.2}$$

and a similar expression for right movers. Using (C.2) we can rewrite (C.1) as

$$\mu e^{\sqrt{k_b - 2}\phi + i\sqrt{k_b}(x_L - x_R)} \tag{C.3}$$

which is precisely the interaction of the bosonic sine-Liouville theory that was conjectured in [51] (see [2]) to be dual to the cigar theory at level k_b.

Using more detailed formulas in [14] it is possible to show that the quotients we have done are consistent with the periodicity conditions for the fields in both theories. In other words we have

$$\begin{aligned} [\mathrm{SU}(2)_{k_s}/\,\mathrm{U}(1)]_{\mathrm{susy}}\,/\,\mathrm{U}(1)_R &= \mathrm{SU}(2)_{k_b}/\,\mathrm{U}(1) \\ (\mathcal{N}=2\quad \text{SineLiouville})_{k_s}\,/\,\mathrm{U}(1)_R &= (\text{Bosonic SineLiouville })_{k_b} \end{aligned} \tag{C.4}$$

So the bosonic version of FZZ follows from [49] by GKO quotienting by the R-symmetry.

D. Computations of winding correlators via T-duality

Let us consider the problem of computing matrix model correlators in the singlet sector. In the region of large λ the matrix model fermion reduces to a relativistic fermion. Since we are at finite temperature this relativistic fermion lives on the circle. If we bosonize this fermion we have a scalar field which is the tachyon. This is a free field in the asymptotic region. We can think of this problem as a conformal field theory, given by the tachyon field or the relativistic fermions, with a boundary. The boundary encodes the small λ region and the reflection amplitudes for the fermions. In the fermion basis the boundary state

[18] In this appendix set $\sqrt{\alpha'} = 1$.

is very simple, we simply have a linear transformation between the left and right moving fermion by a phase which is the scattering phase in the upside down harmonic oscillator potential.

The correlators that we want to compute are most simply expressed by expanding the tachyon field in momentum modes along the euclidean time circle. In the asymptotic region these can be expressed in terms of relativistic fermions. Again, it is convenient to expand the fermions along the euclidean time circle. These will have half integer moding since the fermions are anti-periodic on the euclidean time circle.

For example, if we are interested in the first momentum mode of the tachyon we write it as

$$\alpha_{-1}|0\rangle = \tilde{\psi}_{-1/2}\psi_{-1/2}|0\rangle \tag{D.1}$$

where $|0\rangle$ is the vacuum on the cylinder. We have similar formulas for the incoming and outgoing tachyon modes corresponding to incoming or outgoing fermions. In the cylinder incoming or outgoing fields are related to holomorphic or antiholomorphic fields and they correspond to modes of the tachyon with positive or negative momentum on the circle [52]. So the computation boils down to an expression of the form

$$\langle B|\alpha_{-1}^{in}\alpha_{-1}^{out}|0\rangle\,. \tag{D.2}$$

The boundary state $|B\rangle$ encodes the reflection amplitudes for the fermions. These can be expressed through the identities

$$\begin{aligned}\langle B|\left(\psi_{in,s} - i\mathcal{R}^{-1}(\mu + is/R)\psi_{out,-s}\right) &= 0\,, \qquad s \in Z + \frac{1}{2}\\ \langle B|\left(\tilde{\psi}_{in,s} - i\mathcal{R}(\mu - is/R)\tilde{\psi}_{out,-s}\right) &= 0\end{aligned} \tag{D.3}$$

with the standard anticommutations relations

$$\{\psi_s, \tilde{\psi}_{s'}\} = \delta_{s+s'} \tag{D.4}$$

for both in and out fields. The bounce factors are

$$\mathcal{R}(E) = \sqrt{\frac{\Gamma(\frac{1}{2} - iE)}{\Gamma(\frac{1}{2} + iE)}} \tag{D.5}$$

for the bosonic string.

Using these formulas it is easy to compute the correlator in (D.2). We first express it in terms of fermions through (D.1) and then use (D.3). The net result is

$$\langle B|\alpha_{-1}^{in}\alpha_{-1}^{out}|0\rangle = \langle B||0\rangle\mathcal{R}\left(\mu + i\frac{1}{2R}\right)\mathcal{R}^{-1}\left(\mu - i\frac{1}{2R}\right). \tag{D.6}$$

The overlap $\langle B|0\rangle$ is defined to be the thermal partition function, as computed in [48].

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In summary, (D.6) corresponds to the correlation function for the insertion of a tachyon with momentum one and a tachyon with momentum minus one along the thermal circle. After doing the T-duality, which corresponds to sending $R \to 1/R$ and $\mu \to \mu R$ we obtain the two point correlation function for a tachyon with winding number one and a tachyon with winding number minus one. We identify this with the partition function of the matrix model in the adjoint representations. In this fashion we have derived formula (3.25).

It is clear that we can use this method to compute more complicated correlators of winding operators and hopefully it will be clear to the reader how to proceed.

In this way we could imagine computing the partition function in any representation. All we would need to do is to express the representation in terms of the winding modes. This can be done using the ideas in [53]. We will not go into the details because there is a subtlety that gets in the way. If the decomposition involves modes with different winding number, then the leg factors will be different for these different operators. Since we do not understand the leg factors we cannot confidently assign a representation to a particular configuration of winding correlators. In [54] this subtlety was ignored.

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